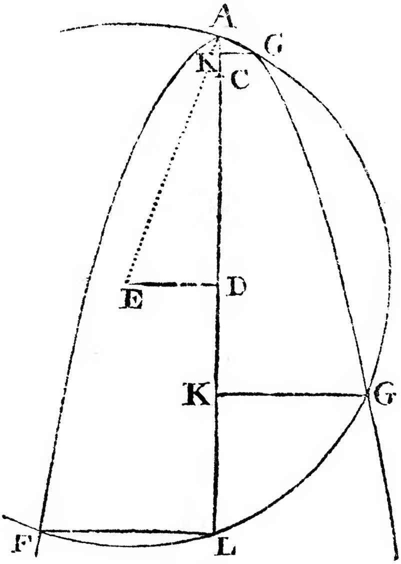
De Conſtructione Problematum Solidorum, ſive Æquationum tertiæ vel quartæ Poteſtatis, unica data Parabola ac Circulo efficienda; diſſertatiuncula: Authore *Edm. Halley*.

QUo pacto æquationes omnes Cubum vel Quadrato-quadratum quantitatis incognitæ involventes, ope Parabolæ cujuſcunq; datæ & Circuli, conſtrui poſſint, clare tradit ac Liquido demonſtrat præclarus ille Carteſius *in Lib. III. Geometriæ ſuæ: ſed primum jubet ſecundum æquationis terminum, ſi adfuerit, tollere, ac deinde reductæ æquationis Radices regula ibidem expoſita elicere. Cum vero operatio iſta nimis laborioſa videatur, nonnullis viſum eſt conſtructionem ſimilem etiam abſq; ulla prævia reductione comminiſci; inter quos* Franciſcus *a* Schooten *Methodum valde facilem ac ſimpliciſſimam pro conſtruendis Cubicis quomodolibet affectis prodidiſſet, ſi modo expoſito principio unde regulam derivavit, Lectoris memoriæ, quam plurimis ac intricatis cautionibus obruit, melius ſtuduiſſet. Nuper vero Vir Cl. D.* Thomas Baker *noſtras, integro libello de conſtructionibus hiſce conſcripto, non ſolum Cubicas ſed etiam Biquadraticas omnes cujuſcunq; generis unica generali regula complexus eſt, eamq; demonſtrationibus ac Exemplis per omnes caſus abunde ſatis illuſtravit; nec non ſub finem modum proponit unde regula iſta generalis inveſtigari poſſit: Haud tamen illum ipſum oſtendit, cujus ope (uti ſuſpicor) Clavem ſuam Geometricam Catholicam obtinuit, vel ſaltem multo facilius obtinere potuit. Cumq; perplexis cautionibus de ſignis + & — Regula hæc D.* Bakeri *non minus obnoxia ſit quam illa* Schooteni, *ut vix abſente libro conſtructiones illas quis tuto peragat; haud injucundum nec Tyronibus incommodum fore vi-*

tum eſt, utriuſq; fundamentum exponere, ac ſimul emendata methodo, in re tam difficili, lucem quantum valeam afferre.

Conſtructio quam tradit Carteſius, *quæq; facillime radices æquationum omnium Cubicarum vel biquadraticarum, ubi deficit ſecundus terminus, eruit, ut nota ſupponi poteſt; attamen cum cardo ſit a quo ſubſequentia pendent, ne diſſertatiuncula hæc capite truncata videatur, ex illius Geometria deſumptam placuit Regulam adjungere, pauculis nonnullis in melius uti reor tranſpoſitis.*

Deficiente ſecundo termino omnes æquationes Cubicæ reducuntur ad hanc formam $z^3 .*. apz. aaq. = 0$, *ac Biquadraticæ ad hanc* $z^4 .*. apzz. aaqz. a^3r = 0$. *(ubi* a *deſignat Latus rectum Parabolæ cujuſvis datæ, quam in Conſtructione adhibere licet.) vel ſumendo* a *pro Unitate, ad hanc* $z^3 .*. pz. q = 0$, *vel ad hanc* $z^4 .*. pzz, qz. r = 0$.



Jam data Parabola F-A G *cujus Axis ſit* A C-D K L *ac latus rectum* a *vel* 1, *fiat* A C *ejus dimidium ac collocetur ſemper a vertice* A *verſus interiora figuræ: dein ſumatur* C D $= \frac{1}{2}$ p *in linea illa* A C *continuata verſus* C *ſi in æquatione fuerit* — p, *vel verſus alteram partem ſi habeatur* + p. *Porro e puncto* D, *aut ex puncto* C *ſi non habeatur quantitas* p, *erigenda eſt ad axem perpendicularis* D-E *æqualis* $\frac{1}{2}$ q, *dextrorſum quidem ſi fuerit* — q, *ad alterum vero axis latus ſi fuerit* + q; *ac Circulus centro* E *radio* A E *deſcriptus, ſi æquatio fuerit tantum Cubica, Parabolam tot punctis* F & G *interſecabit quot veras habet Radices, quarum quidem affirmativæ ut* G K *erunt ad dextram Axis partem, Negativæ ut* F L *ad ſiniſtram.*

Aſt ſi Æquatio Biquadratica fuerit, augeri vel minui debet Circuli Radius A E, *addendo ſi fuerit* — r, *vel ſubducendo, ſi ſit* + r, *ex ejus quadrato rectangulum* a r, *ſeu contentum ſub Latere recto & quantitate data* r; *id quod nullo fere negotio efficitur Geometrice. Hujus vero Circuli interſectiones cum Parabola omnes veras Biquadraticæ Æquationis radices dimiſſis ad Axem perpendiculis exhibebunt; Affirmativas quidem ad dextram Axis, Negativas vero ad ſiniſtram. Totius demonſtrationem* Carteſio *ejus inventori relinqno.*

Notandum hic me operam dare ut ſemper habeantur Radices affirmativæ ad dextrum Axis latus, ut evitetur confuſio a pluribus cautionibus, quarum cauſa minime evidens eſt, neceſſario oritura.

His præmiſſis, ut aditus pateat ad conſtructionem etiam earum æquationum ubi reperitur termiuus ſecundus, conſideranda venit regula pro tollendo termino ſecundo, ac reducenda æquatione ad aliam quæ methodo præcedente conſtrui poſſit. Omnes vero hujus claſſis æquationes cubicæ ad hanc formam $z^3 . b z z . a p z . a a q = 0$, *vel ad hanc* $z^3 . b z z . * . a a q = 0$. *Biquadraticæ vero ad hanc* $z^4 . b z^3 . a p z z . a a q z . a^3 r = 0$, *vel hanc* $z^4 . b z^3 . * . a a q z . a^3 r = 0$, *vel,* $z^4 . b z^3 . a p z z . * . a^3 r = 0$ *vel deniq; ad hanc* $z^4 . b z^3 . * . * . a^3 r = 0$ *reduci poſſunt: e quibus omnibus, prout ſignis* + *&* — *diverſimode connectuntur, ingens oritur varietas; unde Regula generalis omnibus inſerviens obſcura ac maxime difficilis redditur, niſi methodo quam ſubjungimus illuſtrata nodiſq; extricata tractetur.*

Tollitur in Biquadraticis ſecundus terminus, ponendo $x = z + \frac{1}{4} b$, *ſi fuerit* + b *in æquatione, vel* $x = z - \frac{1}{4} b$, *ſi fuerit* — b: *hinc* $x - \frac{1}{4} b$ *in prima caſu, &* $+ \frac{1}{4} b$ *in altero æquatur* z; *& iu æquatione quavis propoſita, ſubſtituta loco* z *quantitate æquali, prodibit nova æquatio termino ſecundo carens, cujus radices omnes* x *data differentia* $\frac{1}{4}$ b *vel excedunt vel deficiunt a radice quæſita* z: *Cum vero in rebus iſtiusmodi plus exempla quam præcepta valere ſolent, proponatur una vel altera æquatio Conſtruenda.*

Exemp. I.

$$z^4 + bz^3 - apzz - aaqz + aaar = 0.$$

Sit $x - \frac{1}{4}b = z$ *Et erit*

$$xx - \frac{1}{2}bx + \frac{1}{16}bb = zz$$

$$xxx - \frac{3}{4}xxb + \frac{3}{16}xbb - \frac{1}{64}bbb = z^3$$

& $$x^4 - bx^3 + \frac{3}{8}bbxx - \frac{1}{16}b^3x + \frac{1}{256}b^4. = z^4.$$

hinc.

$$\begin{array}{lllllll}
x^4 & - bx^3 & + \frac{3}{8}bbxx & - \frac{1}{16}bbbx & + \frac{1}{256}b^4. & = & z^4 \\
 & + bx^3 & - \frac{3}{4}bbxx & + \frac{3}{16}bbbx & - \frac{1}{64}b^4. & = & + bz^3 \\
 & & - apxx & + \frac{1}{2}apbx & - \frac{1}{16}apbb & = & - apzz \\
 & & & - aaqx & + \frac{1}{4}aaqb & = & - aaqz \\
 & & & & + aaar & &
\end{array}$$

Harum omnium ſumma ſit æquatio nova ſecundo termino carens, quæq; proinde juxta regulam Cartesianam conſtrui poſſit, ſumendo loco $\frac{1}{2}p$ *dimidium coefficientis termini tertii per* a *ſive Latus rectum diviſi, hoc eſt* $- \frac{3}{16}\frac{bb}{a} - \frac{1}{2}p$; *ac Loco* $\frac{1}{2}q$, *dimidium coefficientis termini quarti per* aa *diviſi, ſive* $+ \frac{1}{16}\frac{bbb}{aa} + \frac{1}{4}\frac{pb}{a} - \frac{1}{2}q$. *Cujus partes ſigno* + *notatæ ſiniſtrorſum ab Axe, ſigno* — *notatæ dextrorſum collocandæ ſunt, ut habeatur centrum Circuli ad conſtructionem requiſiti, ac cujus interſectiones cum Parabola, dimiſſis in axem perpendiculis, radices omnes veras* x *deſignet, affirmativas quidem ad dextram axis, negativas vero ad ſiniſtram. Cum vero* $x - \frac{1}{4}b = z$, *ducendo lineam Axi parallelam, ad dextrum ejus latus & ad diſtantiam* $\frac{1}{4}b$, *perpendicula illa ad hanc parallelam terminata deſignabunt omnes radices quæſitas* z, *affirmativas ad dextram, negativas vero ad ſiniſtram. Radium circuli quod attinet, habetur ille addendo partes negativas ac auferendo partes affirmativas termini quinti per* a a *diviſi, e quadrato lineæ* A E, *a centro invento* E *ad*

Verticem Parabolæ A *ductæ: id quod maxima ex parte efficitur capiendo loco lineæ* A E *lineam* E O, *quæ ad* O *interſectionem Parabolæ ac parallelæ prædictæ terminatur; ejus enim quadratum omnes termini quinti partes ex ablatione termini ſecundi æquationi novæ ingeſtas complectitur (uti facile probabitur:) ac reſtat ſolummodo ut ipſius* E O *quadratum augeatur, ſi in æquatione habeatur* — r, *vel minuatur ſi ſit* + r, *additione vel ſubductione rectanguli* a r, *unde conflatur quadratum Radii Circuli quæſiti.*

Hæc eſt methodus inveſtigandi regulam centralem Dni Bakeri *omnibus cautionibus libera ac ſatis facilis; ac ſola differentia ex eo provenit, quod ego juxta Axem, ille vero juxta Axi parallelam circuli ejuſdem centrum determinat: quodq; ego ſemper radices affirmativas ex Axis dextro latere invenio, quas ille nunc dextro nunc ſiniſtro conſtituit.*

Æquationes cubicas quod attinet, eæ reduci debent ad Biquadraticas, antequam eadem regula generali conſtrui poſſint; id quod fit ducendo æquationem propoſitam in radicem ſuam z, *unde provenit æquatio Biquadratica in qua deficit terminus ultimus ſive* r: *quapropter ſublato ſecundo termino & invento centro* E, *linea* E O *eſt radius Circuli; cum ſcilicet* a r *ſit* = o, *& in nova æquatione totus terminus quintus ex ipſa ablatione termini ſecundi criatur. Conſtruenda ſit hæc æquatio.*

Exemp. II.

$z^3 - bzz + apz + aaq = 0$: *Quæ ducta in* z *fit*
$z^4 - bz^3 + apzz + aaqz = 0$.
Ad tollendum ſecundum terminum ponatur $x + \frac{1}{4}b = z$, *& fiet*

$$\begin{aligned} x^4 + bx^3 + \tfrac{3}{8}bbxx + \tfrac{1}{16}b^3x + \tfrac{1}{256}b^4 &= +z^4 \\ = bx^3 - \tfrac{3}{4}bbxx - \tfrac{3}{16}b^3x - \tfrac{1}{64}b^4 &= -bz^3 \\ + apxx + \tfrac{1}{2}abpx + \tfrac{1}{16}apbb &= +apzz \\ + aaqx + \tfrac{1}{4}aaqb &= +aaqz \end{aligned}$$

In hac nova Æquatione, tertii termini ſemicoefficiens per a *diviſa*, viz. $= \frac{3bb}{16a} + \frac{1}{2}p$, *loco* $\frac{1}{2}p$ *uſurpanda eſt; ac coefficientis*

cientis termini quarti dimidium, diviſum per aa *Lateris recti quadratum,* viz. $-\frac{bbb}{16aa}+\frac{pb}{4a}+\frac{1}{2}q$, *vicem ipſius* $\frac{1}{2}q$ *in conſtructione* Carteſii *ſubit; unde centrum* E *determinatur. Deinde ducta Axi parallela ad diſtantiam* $\frac{1}{4}b$ *ad ſiniſtrum ejus latus (ob* $x+\frac{1}{4}b=z$*) cujus interſectio cum Parabola ſit* O; *circulus centro* E, *Radio* E O *deſcriptus Parabolam ſecet vel tanget in tot punctis quot æquatio veras habet radices: quæ quidem radices ſeu* z *ſunt perpendicula de punctis illis in Axi parallelam demiſſa; ad dextram-quidem Affirmativæ, Negativæ ad ſiniſtram.*

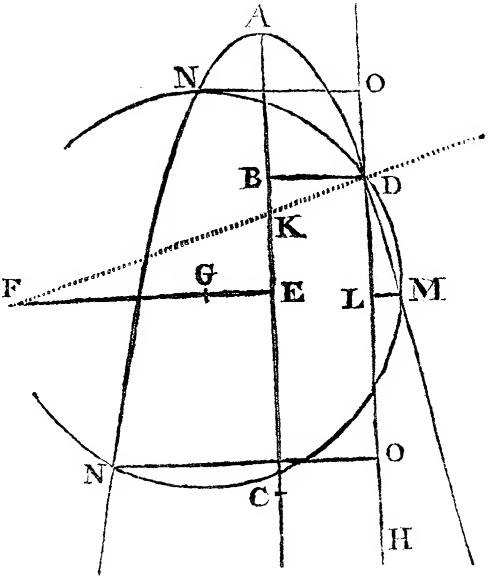
Si in æquatione defuerit terminus tertius vel quartus vel uterq;, in inveſtiganda regula centrali nulla omnino obſervanda eſt methodus differentia, ſed deficiente quantitate p *vel* q, *deerunt partes illæ linearum* C D *ac* D E *ex quantitate illa aliquo modo deductæ, ac procedendum eſt cum reliquis coefficientibus termini tertii and quarti in æquatione nova, ſicut in præmiſſis exemplis præſcriptum eſt.*

Hactenus Cl. Bakeri *methodum generalem pertractavimus, qua quidem nulla alia facilior ac paratior expectanda eſt, aſſumpta ad conſtructionem ſive Parabola, ſive alia quævis linea curva, cum ſcilicet æquatio ad Biquadraticam aſcendit. Etenim dum hæc ſcribo mihi occurrit regulæ Centralis Effectio Geometrica præter omnem ſpem expedita, ac harum rerum Curioſis abunde ſatisfactura.*

Deſcripta Parabola N A M, *cujus vertex* A, *Axis* A B C *ac latus rectum* a, *reducatur æquatio ad hanc formam* $z^4.\,bz^3.\,apzz.\,aaqz.\,a^3r.=0$ *vel ad hanc* $z^3.\,bzz.\,apz.\,aaq=0$ *ſi cubica tantum fuerit: dein ad diſtantiam* B D $=\frac{1}{4}b$ *ducatur linea* D H *Axi parallela, ad ſiniſtram quidem ſi fuerit* $-b$, *ad dextram ſi* $+b$, *parabolæ occurrens in puncto* D; *de quo dimittatur perpendiculum in axem* B D. *In linea* A B *continuata verſus* B *fiat* B K $=\frac{1}{2}a$, *& ducatur linea* D K *utrinq; interminata. Porro ſit* K C $=2$ A B *in Axe ſemper ultra* K *continuato; ac ſi habeatur quantitas* p *ſigno* $-$ *affecta, verſus eaſdem partes etiam ſnmatur* C E $=\frac{1}{2}p$, *vel in contrarias,*

ſi

ſi habeatur $+p$, *ac e puncto* E *erigatur Axi perpendiculum* E F *(vel e puncto* C *ſi defuerit quantitas* p*) lineæ* D K, *ſi opus eſt continuatæ, occurrens in puncto* F; *quod quidem circuli requiſiti centrum eſt, ſi defuerit quantitas* q; *Aſt ſi habeatur* q, *ſumenda eſt in* F E, *ſi opus eſt continuata, linea* $FG = \frac{1}{2}q$, *ſiniſtrorſum quidem ſi fuerit* $+q$, *dextrorſum ſi* $-q$ *collocanda: Et punctum* G *erit centrum circuli ad conſtructionem propoſitam idonei; ejuſq; Radius, ſi defuerit quantitas* r, *hoc eſt ſi tantum cubica fuerit, erit linea* G D; *cujus quadratum in Biquadraticis augendum eſt, ſi fuerit* $-r$, *vel minuendum ſi* $+r$ *additione vel ſubductione rectanguli ſub* r *& latere recto. Deſcripto ſic Circulo, ab interſectionibus ejus cum Parabola demiſſis in lineam* D H *perpendiculis, quæ ad ſiniſtram ſunt, ut* N O, *radices æquationis negativas ſemper deſignant, quæ ad dextram ut* M L *affirmativas.*



Aliter ac paulo ſimplicius Æquationes cubicæ juxta Schooteni *Regulam conſtruuntur, quaq; etiam radices ad Axem referuntur: quoniam vero ipſe inventor nec modum inveniendi nec demonſtrationem inventi exponit, non abs re erit ejuſdem fundamentum hic adjicere, ſimul atq; Effectionem Geometricam concinniorem reddere, atq; cautionibus quibus implicatur extricare.*

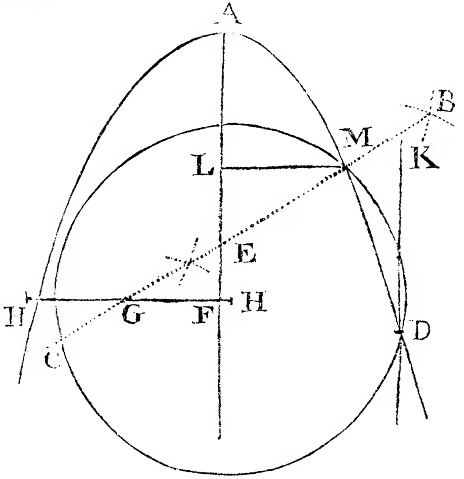
Hæc Regula derivatur ex eo quod omnis æquatio Cubica reduci poſſit ad Biquadraticam, in qua deficiet terminus ſecundus: Hoc fit ducendo æquationem propoſitam in $z - b = 0$, *ſi fuerit* $+b$ *in* *æquatione, vel in* z + b = o, *ſi fuerit* — b; & *æquatio nova producta eaſdem habebit radices cum Cubica, atq, inſuper alteram ipſi* — b *æqualem, ſi fuerit* — b *in æquatione, vel contra.*

Proponatur conſtruenda $z^3 - z^2 b + apz + aaq = 0.$ *Hæc ducta in* z + b *fit*

$$\begin{aligned} &z^4 - z^3 b + apz^2 + aaqz \\ &\quad + z^3 b - bbzz + abpz + aaqb. \end{aligned}$$

Hic deficit ſecundus terminus, ac coefficiens tertii $-bb + ap$ *dat* $-\frac{bb}{2a} + \frac{1}{2}p$ *loco* $\frac{1}{2}p$ *vel* C D *in Conſtructione* Carteſii, & *ex dimidio coefficientis termini quarti fit* $+\frac{1}{2}q + \frac{bp}{2a}$ *loco* $\frac{1}{2}q$ *vel* D E *uſurpanda; adeoq; determinatur centrum circuli quæſiti: atq; ob datam unam ex radicibus æquationis novæ*, viz. — *vel* + b, *dabitur etiam punctum in circumferentia, id eſt Radius ejus. Deniq; deſcripto circulo, ab interſectionibus ejus cum Parabola demiſſa in Axem perpendicula Æquationis radices exhibebunt, affirmativas & negativas, eadem lege ac ſupra.*

Inveſtigatur autem centrum Circuli conſtructione qerquam facili, cæteriſq; omnibus in Cubicis præferenda. Deſcriptæ Parabolæ A M D *ſit vertex* A, *atq; Axis* A F: *ad diſtantiam ipſi* b *æqualem ducatur Axi parallela* D K, *ad dextram ſi fuerit* + b *in Æquatione, ad ſiniſtram ſi* — b, *quæ Parabolæ occurrat in puncto* D. *Centris* D & A *deſcribantur radiis æqualibus arcus occulti utrinq; ſeſe interſecantes, ac per ſectionum puncta ducatur linea interminata* B C, *quæ medio lineæ ſuppoſitæ* A D *perpendiculariter inſiſtat,* & *Axi occurrat in puncto* E. *Ab* E, *inferne quidem ſi in æquatione habeatur* = p, *vel ſuperne verſus* A *ſi fuerit* + p, *ponatur*



tur $EF = \frac{1}{2}p$; *& ex* F *(vel ex* E *ſi defuerit* p*) educatur perpendiculum* F G, *lineæ* B C *occurrens in puncto* G; *& in* G F *producta fiat* $GH = \frac{1}{2}q$, *dextrorſum quidem ſi in æquatione habeatur* $-q$, *aliter ſiniſtrorſum, applicanda : ac punctum* H *erit centrum quæſitum,* H D *vero circuli Radius, qui demiſſis in axem perpendiculis ab interſectionibus ſuis cum Parabola, ut* L M, *Radices omnes, ut prius, commonſtrabit. Quomodo vero conſtructio hæc ex præmiſſis conſequatur, per ſe ſatis edens eſt, nec opus eſt ut in eadem demonſtranda diutius immorer.*

Ne in his edendis fruſtraneam navaſſe operam, & ex aliorum inventis gloriolam captare videar, conſulat Lector Cl. Bakeri *librum Anno* 1684 Londini *editum, & quæ de hoc Argumento ſcripſit* a Schooten *in Commentario ſuo in Librum III. Geometriæ Carteſianæ. Brevi conceſſo otio tractatulum alium de numero Radicum in hujuſmodi Æquationibus, earumq; limitibus, ex contemplatione Conſtructionum paæcedentium, aggredi ac in lucem proferre ſtatuo.*
